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WAVE MOTION OF AN IDEAL FLUID IN A NARROW OPEN CHANNEL

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This paper considers a nonlinear integrodifferential model constructed for the motion of an ideal incompressible fluid in an open channel of variable section using the long-wave approximation. A characteristic equation for describing the perturbation propagation velocity in the fluid is derived. Necessary and sufficient conditions of generalized hyperbolicity for the equations of motion are formulated, and the characteristic form of the system is calculated. In the case of a channel of constant width, the model reduces to the Riemann integral invariants which are conserved along the characteristics. It is found that, during the evolution of the flow, the type of the equations of motion can change, which corresponds to long-wave instability for a certain velocity distribution along the channel width.

Key words: *long-wave approximation, horizontal shear flows, integrodifferential equations, hyperbolicity.*

Introduction. The modeling of nonlinear wave motion on the surface of a thin fluid layer is an important fundamental and applied problem which has been the subject of many papers, including [1–3] and others. The classical one-dimensional shallow-water equations are used to describe fluid wave motion in open channels with straight lateral boundaries. Applications often involve a need to model fluid motion in channels with curved walls, resulting in a more complex, substantially two-dimensional mathematical formulations. Most of the mathematical models of plane–parallel motion of liquids and gases in the long-wave approximation reduce to nonlinear integrodifferential equations. A qualitative analysis of some models of long-wave theory was performed in [4] on the basis of the generalization of the hyperbolicity concept and the method of characteristics for equations with operator coefficients which were developed by Teshukov [5]. A known example of using the method of generalized characteristics [5, 6] is the model of eddying shallow water [7] which describes plane–parallel motion of a free-boundary thin layer of an ideal fluid with a nontrivial velocity distribution along the flow depth.

The mathematical model proposed in the present paper for fluid motion in an extended channel with curved lateral walls using mixed Eulerian–Lagrangian variables belongs to the class of systems with operator coefficients, to which the Teshukov method applies [5]. This allows one to calculate the perturbation propagation velocity (the model has a discrete and continuous characteristic spectrum) and to formulate necessary and sufficient hyperbolicity conditions for the equations of motion. The validity of the hyperbolicity condition is verified for a nonstationary exact solution. It is shown that long-wave instability can occur during the evolution of horizontal shear flow in a channel.

1. Long-Wave Model. We consider the spatial motion of an ideal incompressible fluid with free boundary $z = h(t, x, y)$ in an open channel with even bottom $z = 0$ and lateral walls $y = Y_1(x)$ and $y = Y_2(x)$ in a gravity field. The equations of motion in dimensionless variables are written as

$$\begin{gathered} u_t + (\boldsymbol{v} \cdot \nabla)u + p_x = 0, \qquad \varepsilon^2(v_t + (\boldsymbol{v} \cdot \nabla)v) + p_y = 0, \\ \varepsilon^2(w_t + (\boldsymbol{v} \cdot \nabla)w) + p_z = -g, \qquad \operatorname{div} \boldsymbol{v} = 0. \end{gathered} \tag{1}$$

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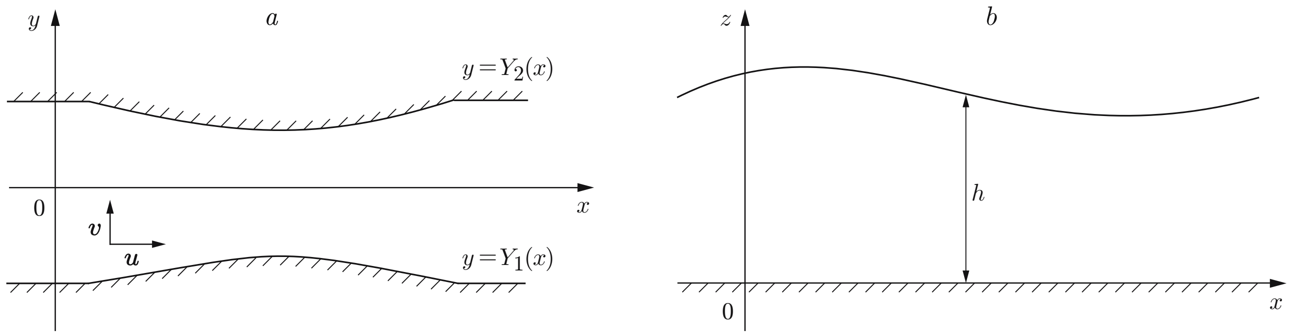

Fig. 1. Fluid flow in an open channel of variable section: (a) section of the channel by the plane $z = \text{const}$; (b) section of the channel by the plane $y = \text{const}$.

On the free boundary $z = h(t, x, y)$, the following kinematic and dynamic boundary conditions should be satisfied:

$$h_t + uh_x + vh_y - w = 0, \qquad p = p_0; \tag{2}$$

on the even bottom $z = 0$ and lateral boundaries $y = Y_i(x)$ $(i = 1, 2)$, the nonpenetration conditions are imposed:

$$w = 0, \qquad uY_i'(x) = v. \tag{3}$$

Here $u = L^{-1}T\bar{u}$, $v = l^{-1}T\bar{v}$, $w = l^{-1}T\bar{w}$, $p = \rho^{-1}L^{-2}T^2\bar{p}$, $x = L^{-1}\bar{x}$, $y = l^{-1}\bar{y}$, $z = l^{-1}\bar{z}$, and $t = T^{-1}\bar{t}$ are the components of the velocity, pressure, Cartesian coordinates, and time, respectively; $\bar{\boldsymbol{v}} = (\bar{u}, \bar{v}, \bar{w})$, $\bar{p}$, $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y}, \bar{z})$ and $\bar{t}$ are the corresponding dimensional variables. The quantity L specifies the characteristic scale on the x axis directed along the channel, and the quantity l on the y and z axes (Fig. 1); $T = L/\sqrt{al}$ is the characteristic time scale (a has the dimension of acceleration); the constants ρ and g are the density of the fluid and the dimensionless acceleration due to gravity, ∇ and div are gradient and divergence operators calculated from the spatial variables, and $\varepsilon = l/L$ is a dimensionless small parameter.

The equations of the approximate model describing long-wave propagation in a narrow channel are obtained from Eqs. (1) by passing to the limit $\varepsilon \to 0$. In this case, the law of conservation of the vertical momentum component [the third equation of system (1)] implies a hydrostatic pressure distribution $p = g(h - z) + p_0$, where the constant p_0 is the pressure on the free boundary. Simple transformations of Eqs. (1) taking into account the boundary conditions on the free boundary (2) and on the motionless walls (3) lead to the system of equations

$$\begin{gathered} u_t + uu_x + vu_y + wu_z + gh_x = 0, \qquad h_y = 0, \qquad w = -\int_0^z (u_x + v_y)\,dz, \\ h_t + \Big(\int_0^h u\,dz\Big)_x + \Big(\int_0^h v\,dz\Big)_y = 0, \qquad uY_i'(x) - v\Big|_{y=Y_i} = 0, \end{gathered} \tag{4}$$

which describes the motion of an ideal fluid in a narrow open channel in the long-wave approximation.

Let us consider the class of flows in which the horizontal velocity components u and v do not depend on the vertical coordinate z. In this case, the long-wave model (4) becomes

$$\begin{gathered} u_t + uu_x + vu_y + gh_x = 0, \qquad h_y = 0, \\ h_t + (uh)_x + (vh)_y = 0, \qquad uY_i'(x) - v|_{y=Y_i} = 0. \end{gathered} \tag{5}$$

We note that, in the case of straight lateral boundaries $Y_i = \text{const}$ and zero velocity component $(v \equiv 0)$, model (5) reduces to the classical one-dimensional shallow-water equations [1]. The present paper considers flows in channels with curved walls, leading to the necessity of investigating substantially two-dimensional motion described by Eqs. (5).

To study the mathematical properties of Eqs. (5), it is convenient to transform to semi-Lagrangian coordinates by the change of the variable $y = \Phi(t, x, \lambda)$, where the function Φ is a solution of the Cauchy problem [8]

$$\Phi_t + u(t, x, \Phi)\Phi_x = v(t, x, \Phi), \qquad \Phi\Big|_{t=0} = \lambda Y_2(x) + (1 - \lambda)Y_1(x). \tag{6}$$

The Lagrangian variable $\lambda \in [0,1]$; the values $\lambda = 0$ and $\lambda = 1$ correspond to the lateral boundaries of the channel $y = Y_1(x)$ and $y = Y_2(x)$. The equations $y = \Phi(t,x,\lambda_0)$, where $\lambda_0 = \text{const}$, define real surfaces consisting of the same particles. In the new variables, the functions $u(t,x,\lambda)$ and $H(t,x,\lambda) = h\Phi_\lambda$ are described by the integrodifferential system of equations

$$u_t + uu_x + gh_x = 0, \qquad H_t + (uH)_x = 0, \qquad h = \frac{1}{\eta}\int_0^1 H\,d\lambda, \tag{7}$$

where $\eta(x) = Y_2(x) - Y_1(x) > 0$ is the given channel width. Below, Eqs. (7) are derived as a consequence of the long-wave model (5).

We use $\tilde{u}$ and $\tilde{v}$ to denote the horizontal velocity components in semi-Lagrangian coordinates:

$$\tilde{u}(t,x,\lambda) = u(t,x,\Phi(t,x,\lambda)), \qquad \tilde{v}(t,x,\lambda) = v(t,x,\Phi(t,x,\lambda)).$$

Then, the derivatives are represented as

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial \Phi}{\partial x}, \qquad \frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \Phi}{\partial x}, \qquad \frac{\partial \tilde{u}}{\partial \lambda} = \frac{\partial \Phi}{\partial \lambda}\frac{\partial u}{\partial y}. \tag{8}$$

Relations (8) and (6) imply the obvious equalities

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\,\frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\Big(\frac{\partial \Phi}{\partial t} + u\,\frac{\partial \Phi}{\partial x}\Big) = \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -g\,\frac{\partial h}{\partial x}$$

which lead to the first equation of system (7). By virtue of formulas (8) and $h_y = 0$, the third equation of system (5) becomes

$$0 = \frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} + h\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) = \frac{\partial h}{\partial t} + \frac{\partial\,(\tilde{u}h)}{\partial x} + \frac{h}{\Phi_\lambda}\Big(\frac{\partial \tilde{v}}{\partial \lambda} - \frac{\partial \tilde{u}}{\partial \lambda}\frac{\partial \Phi}{\partial x}\Big). \tag{9}$$

Differentiation of the first equation in (6) with respect to the variable λ yields

$$\Phi_{\lambda t} + \tilde{u}\Phi_{\lambda x} = \frac{\partial \tilde{v}}{\partial \lambda} - \frac{\partial \tilde{u}}{\partial \lambda}\frac{\partial \Phi}{\partial x}. \tag{10}$$

Combining (9) and (10) and setting $\Phi_\lambda \neq 0$, we obtain the equation

$$\frac{\partial\,(h\Phi_\lambda)}{\partial t} + \frac{\partial\,(\tilde{u}h\Phi_\lambda)}{\partial x} = 0,$$

which for $H(t,x,\lambda) = h(t,x)\Phi_\lambda$ coincides with the second equation of system (7). Since $\Phi\Big|_{\lambda=0} = Y_1(x)$ and $\Phi\Big|_{\lambda=1} = Y_2(x)$, the obvious equality

$$\eta(x) = Y_2(x) - Y_1(x) = \int_0^1 \frac{\partial \Phi}{\partial \lambda}\,d\lambda = \frac{1}{h(t,x)}\int_0^1 H\,d\lambda$$

leads to the last equation of the integrodifferential model (7).

Transformation to the semi-Lagrangian coordinates is a reversible change of variables if $\Phi_\lambda \neq 0$. Indeed, let the functions $u(t,x,\lambda)$ and $H(t,x,\lambda)$ be found. Then, the fluid layer depth $h(t,x)$ is known, and the Eulerian coordinate y and the velocity component v can be obtained from the formulas

$$y = \Phi(t,x,\lambda) = Y_1(x) + \frac{1}{h}\int_0^\lambda H\,d\lambda, \qquad v = \Phi_t + u\Phi_x.$$

We note that in the case of constant channel width ($\eta = \text{const}$), model (7) coincides with the eddying shallow water equations describing plane–parallel motion of a free-boundary ideal fluid in a gravity field with a nontrivial velocity distribution with depth [5, 6].

Let us consider solutions of system (7) that satisfy the inequality $H > 0$. This inequality implies that the fluid depth in the channel h and the Jacobian of the transformation to the semi-Lagrangian variables $\Phi_\lambda(t,x,\lambda)$ is larger than zero. In this case, the Lagrangian variable λ increases with increasing Eulerian coordinates y. The characteristic properties of the model are analyzed under the assumption of a monotonic change in the velocity $u(t, x, \lambda)$ along the channel width. For definiteness, we set $u_\lambda(t, x, \lambda) > 0$. If this condition is satisfied at the initial time $t = 0$, then, by virtue of system (7), it is satisfied for all $t > 0$. The above statements are also valid for flows symmetric about the central line of the channel $y = 0$ [in this case, $Y_2(x) = -Y_1(x)$] that satisfy the conditions $u(t, x, \lambda) = u(t, x, 1 - \lambda)$, and $u_\lambda \neq 0$ for $\lambda \in (0, 1/2)$. In this case, the line $y = 0$ (or $\lambda = 1/2$) can be considered an impenetrable boundary and the flow can be examined in the region $Y_1 < y < 0$ (or $0 < y < Y_2$). In addition, in view of the results of [9], the hyperbolicity conditions for the equations of motion for flows with a nonmonotonic velocity distribution $u(t, x, \lambda)$ along the channel width can be formulated under the assumption that

$$u_\lambda > 0 \quad \text{for} \quad 0 < \lambda < \lambda_*(t, x), \qquad u_\lambda < 0 \quad \text{for} \quad \lambda_*(t, x) < \lambda < 1,$$
$$u_{\lambda\lambda}(t, x, \lambda_*(t, x)) \neq 0, \qquad u(t, x, 0) < u(t, x, 1). \tag{11}$$

2. Generalized Characteristics of the Equations of Motion. The integrodifferential model (7) belongs to the class of systems of equations with operator coefficients for which the notions of characteristics and hyperbolicity were proposed [5]. System (7) can be represented as

$$\boldsymbol{U}_t + A\langle \boldsymbol{U}_x \rangle = \boldsymbol{G}, \tag{12}$$

where

$$\boldsymbol{U} = (u, H)^{\mathrm{t}}$$

is the vector of the required quantities,

$$\boldsymbol{G} = \Big(\frac{g\eta'(x)}{\eta^2} \int_0^1 H \, d\lambda, \ 0 \Big)^{\mathrm{t}}$$

is the right side of Eq. (12), and

$$A\langle \boldsymbol{U}_x \rangle = \Big(u u_x + \frac{g}{\eta} \int_0^1 H_x \, d\lambda, \ H u_x + u H_x \Big)^{\mathrm{t}}$$

is the result of the action of the matrix operator A on the vector function $\boldsymbol{U}_x$. According to [5], the characteristic curve of system (12) is defined by the differential equation $x'(t) = k(t, x)$, in which the propagation velocity of the characteristic k is an eigenvalue of the problem

$$(\boldsymbol{F}, (A - kI)\langle \boldsymbol{\varphi} \rangle) = 0. \tag{13}$$

The solution of Eq. (13) for the functional $\boldsymbol{F} = (F_1, F_2)$ is sought in the class of locally integrable or generalized functions. The functional $\boldsymbol{F}$ acts on the variable λ, and t and x are treated as parameters, I is an identical map, and $\boldsymbol{\varphi}$ is a trial smooth vector function with the components $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$. Applying the functional $\boldsymbol{F}$ to Eq. (12), we obtain the following relation on the characteristic:

$$(\boldsymbol{F}, \boldsymbol{U}_t + k\boldsymbol{U}_x) = (\boldsymbol{F}, \boldsymbol{G}). \tag{14}$$

System (12) is generalized hyperbolic [5] if all eigenvalues k are real and the set of relations on the characteristics (14) is equivalent to the initial equations (12), i.e., the system of eigenfunctionals is complete in the space considered.

For the long-wave equations of vertical shear flows and for some other integrodifferential models are close in structure to Eqs. (7), the characteristic properties are investigated in [4–6]. Therefore, below we give only a brief solution of problem (13) and formulate (without proof) the hyperbolicity conditions for model (7).

2.1. *Eigenfunctionals and Relations on the Characteristics.* Taking into account the independence of the components of the trial vector function $\boldsymbol{\varphi}$, from Eqs. (13) we obtain the equality

$$(F_1, (u - k)\varphi_1) + (F_2, H\varphi_1) = 0, \qquad g\eta^{-1} \int_0^1 \varphi_2 \, d\lambda (F_1, 1) + (F_2, (u - k)\varphi_2) = 0. \tag{15}$$

By virtue of the first equation in (15), the action of the functional F_2 on an arbitrary smooth function ψ can be expressed in terms of F_1:

$$(F_2, \psi) = -(F_1, (u-k)H^{-1}\psi).$$

Using this expression, from the second formula in (15) we have

$$(F_1, (u-k)^2 H^{-1}\psi) - \frac{g}{\eta}\int_0^1 \psi\, d\lambda (F_1, 1) = 0. \tag{16}$$

Let us consider the set of numbers k belonging to the complex plane, except for the segment $[u_0, u_1]$ [u_0 and u_1 are the minimum and maximum values of $u(t, x, \lambda)$ depending on the variable λ for fixed t and x]. Then, Eq. (16) can be written in equivalent form

$$(F_1, \psi) = \Big(\frac{g}{\eta}\int_0^1 \frac{\psi H\, d\lambda}{(u-k)^2}\Big)(F_1, 1).$$

For $\psi = 1$, this equation implies the condition of the existence of nontrivial solutions of problem (15) — the characteristic equation for the perturbation propagation velocity k:

$$\chi(k) = 1 - \frac{g}{\eta}\int_0^1 \frac{H\, d\lambda}{(u-k)^2} = 0. \tag{17}$$

If $k = k^i$ is a root of Eq. (17), problem (15) has a nontrivial solution $\boldsymbol{F}^i = (F_1^i, F_2^i)$. The action of the functionals F_1^i and F_2^i is given by the formulas

$$(F_1^i, \psi) = \frac{g}{\eta}\int_0^1 \frac{H\psi\, d\lambda}{(u-k^i)^2}, \qquad (F_2^i, \psi) = -\frac{g}{\eta}\int_0^1 \frac{\psi\, d\lambda}{u-k^i}.$$

The functionals $\boldsymbol{F}^i$ corresponding to the eigenvalues k^i can be written as

$$(\boldsymbol{F}^i, \boldsymbol{\varphi}) = \int_0^1 \boldsymbol{f}^i \cdot \boldsymbol{\varphi}\, d\lambda = \int_0^1 (f_1^i\varphi_1 + f_2^i\varphi_2)\, d\lambda, \tag{18}$$

where f_1^i and f_2^i are integrable functions. From formula (13), it follows that $\boldsymbol{f}^i = (f_1^i, f_2^i)$ is an eigen vector function of the conjugate operator A^* which is linked to the operator A by the relation $(A\boldsymbol{f}, \boldsymbol{g}) = (\boldsymbol{f}, A^*\boldsymbol{g})$. Here $(\boldsymbol{f}, \boldsymbol{g})$ is the scalar product in $L_2[0, 1]$.

Completing the construction of solutions of Eqs. (15), we consider the segment $[u_0, u_1]$ under the assumption that the velocity $u(t, x, \lambda)$ is monotonic along the channel width: $u_\lambda(t, x, \lambda) > 0$, $u_0 = u(t, x, 0)$, and $u_1 = u(t, x, 1)$. Below, it is proved that any point of the segment $[u_0, u_1]$ belongs to the characteristic spectrum of problem (15), but the eigenfunctionals are generalized eigen functions of the conjugate operator A^* and are not expressed in the form of (18). Let $k = u(t, x, \lambda)$ (λ be any fixed value in the interval $[0, 1]$). In this case, system (15) becomes

$$\begin{gathered}\Big(F_1, (u(\nu) - u(\lambda))\varphi_1(\nu)\Big) + \Big(F_2, H(\nu)\varphi_1(\nu)\Big) = 0,\\ g\eta^{-1}\int_0^1 \varphi_2(\nu)\, d\nu\, (F_1, 1) + \Big(F_2, (u(\nu) - u(\lambda))\varphi_2(\nu)\Big) = 0.\end{gathered} \tag{19}$$

Here the functionals act over the variable ν. For brevity, we use the notation $f(\nu) = f(t, x, \nu)$, $f(\lambda) = f(t, x, \lambda)$. It is easy to see that the functionals $\boldsymbol{F}^{1\lambda} = (F_1^{1\lambda}, F_2^{1\lambda})$ and $\boldsymbol{F}^{2\lambda} = (F_1^{2\lambda}, F_2^{2\lambda})$, whose action is given by the formulas

$$\begin{gathered}(F_1^{1\lambda}, \psi(\nu)) = -\psi'(\lambda), \qquad (F_2^{1\lambda}, \psi(\nu)) = \frac{u_\lambda}{H}\,\psi(\lambda),\\ (F_1^{2\lambda}, \psi(\nu)) = \psi(\lambda) + \frac{g}{\eta}\int_0^1 \frac{H(\nu)(\psi(\nu) - \psi(\lambda))\, d\nu}{(u(\nu) - u(\lambda))^2}, \qquad (F_2^{2\lambda}, \psi(\nu)) = -\frac{g}{\eta}\int_0^1 \frac{\psi(\nu)\, d\nu}{u(\nu) - u(\lambda)}\end{gathered} \tag{20}$$

are a solution of Eqs. (19). In (20), the improper integrals are understood in the sense of the principal value. These functionals are generalized functions. In particular, $F_1^{1\lambda} = \delta'(\nu - \lambda)$ and $F_2^{1\lambda} = u_\lambda H^{-1}\delta(\nu - \lambda)$, where $\delta(\nu - \lambda)$ and $\delta'(\nu - \lambda)$ is the Dirac delta function and its derivative.

Applying the eigenfunctionals $\boldsymbol{F}^i$, $\boldsymbol{F}^{1\lambda}$, and $\boldsymbol{F}^{2\lambda}$ to system (12), we obtain the following relations on the characteristics [the characteristic form of Eqs. (7)]

$$R_t + uR_x = \frac{g\eta'(x)}{\eta^2(x)} \int_0^1 \frac{u(t,x,\nu)H(t,x,\nu)\,d\nu}{u(t,x,\nu) - u(t,x,\lambda)}, \qquad \omega_t + u\omega_x = 0,$$
$$r_t^i + k^i r_x^i = \frac{g\eta'(x)}{\eta^2(x)} \int_0^1 \frac{u(t,x,\lambda)H(t,x,\lambda)\,d\lambda}{u(t,x,\lambda) - k^i(t,x)}. \tag{21}$$

Here

$$R(t,x,\lambda) = u(t,x,\lambda) - \frac{g}{\eta(x)} \int_0^1 \frac{H(t,x,\nu)\,d\nu}{u(t,x,\nu) - u(t,x,\lambda)}, \qquad \omega(t,x,\lambda) = \frac{u_\lambda(t,x,\lambda)}{H(t,x,\lambda)},$$

$$r^i(t,x) = k^i(t,x) - \frac{g}{\eta(x)} \int_0^1 \frac{H(t,x,\lambda)\,d\lambda}{u(t,x,\lambda) - k^i(t,x)}.$$

In the case of a channel of constant width, the right sides Of Eqs. (21) vanish and the quantities R, ω, and r^i introduced above are conserved in differentiation along the characteristics, i.e., they are Riemann integral invariants.

2.2. *Conditions of Generalized Hyperbolicity for the Equations of Motion.* We investigate the properties of the characteristic function $\chi(k)$ on the real axis. The derivative

$$\chi'(k) = -\frac{2g}{\eta} \int_0^1 \frac{H\,d\lambda}{(u-k)^3}$$

takes negative values for $k < u_0$ and is a positive function for $k > u_1$. If $k \to \pm\infty$, $\chi(k) \to 1$; if $k \to u_0$ or $k \to u_1$, $\chi(k) \to -\infty$ (the integral in expression (17) diverges if $0 < a < H$ and the function u is continuously differentiable with respect to λ on the interval $[0,1]$). In this case, the function $\chi(k)$ is convex upward since the second derivative $\chi''(k) < 0$. From the above-mentioned properties of the characteristic function, it follows that Eq. (17) has two real roots k^1 and k^2 outside the segment $[u_0, u_1]$; $k^1(t,x) < u_0(t,x)$, and $k^2(t,x) > u_1(t,x)$. If the solution has singularities or if u_λ tends to infinity for $\lambda = 0$ ($\lambda = 1$), the integral in (17) can become divergent for $k = u_0$ (or for $k = u_1$). This can lead to the disappearance of one of the characteristic roots. Then, in the corresponding direction, the wave perturbations move together with the flow. In the following, we consider smooth solutions of Eqs. (7) for which Eq. (17) always has two real roots k^1 and k^2 outside the segment $[u_0, u_1]$.

In addition to the real roots, the characteristic equation (17) can have complex roots. We show that if for the examined solution of Eqs. (7) there is a complex characteristic root $k = k_0 + ik_1$ ($k_1 \neq 0$), it belongs to some subregion of the circle $|k - (u_1 + u_0)/2| \leq (u_1 - u_0)/2$, by analogy with the Howard semicircle theorem. Separating the real and imaginary parts in (17), we obtain the relations

$$1 - \frac{g}{\eta} \int_0^1 \frac{((u-k_0)^2 - k_1^2)H\,d\lambda}{|u-k|^4} = 0, \qquad k_1 \int_0^1 \frac{(u-k_0)H\,d\lambda}{|u-k|^4} = 0. \tag{22}$$

Introducing the notation $r = (u_1 - u_0)/2$ and $r_0 = (u_1 + u_0)/2$ and using the identity

$$(u - r_0)^2 = (u-k_0)^2 + (k_0 - r_0)^2 + 2(k_0 - r_0)(u - k_0),$$

the second equation of (22) for $k_1 \neq 0$, and the third equation of (7), we reduce the first equation in (22) to the form

$$\int_0^1 \frac{|u-k|^4 h^{-1} + gk_1^2 - g(u-r_0)^2 + g(k_0 - r_0)^2}{|u-k|^4}\,H\,d\lambda = 0.$$

By virtue of the obvious inequalities $(u-r_0)^2 \le r^2$ and $|u-k|^4 \ge k_1^4$ and the condition $H>0$, we have

$$\Big(k_1^4 h^{-1} + g k_1^2 + g(k_0-r_0)^2 - g r^2\Big)\int_0^1 \frac{H\,d\lambda}{|u-k|^4} \le 0.$$

Therefore, any complex characteristic root belongs to the region $(gh)^{-1}k_1^4 + k_1^2 + (k_0-r_0)^2 \le r^2$ of the circle $|k-r_0|\le r$. From this it follows that, if a complex characteristic root k appears during the evolution of the flow, it is separated from the segment of the continuous characteristic spectrum $[u_0,u_1]$ at the time of origin.

The conditions of no complex roots of the characteristic equation (17) are formulated in terms of the analytical function $\chi(z)$, or more precisely, its limiting values from the upper χ^+ and lower χ^- complex half-planes on the segment $[u_0,u_1]$:

$$\chi^\pm(u(\lambda)) = 1 + \frac{g}{\eta}\Big(\frac{1}{\omega_1(u_1-u(\lambda))} - \frac{1}{\omega_0(u_0-u(\lambda))} - \int_0^1 \Big(\frac{1}{\omega(\nu)}\Big)_\nu \frac{d\nu}{u(\nu)-u(\lambda)} \mp \frac{\pi i}{u_\lambda}\Big(\frac{1}{\omega}\Big)_\lambda\Big). \qquad (23)$$

Here $\omega = u_\lambda/H$; the subscripts 0 and 1 correspond to the values of the functions for $\lambda=0$ and $\lambda=1$; i is an imaginary unit. The limiting values of the function $\chi(z)$ are calculated using the Sokhotsky–Plemelj formulas [10].

The lemmas given below formulate the conditions of no complex characteristic roots and the completeness of the system of eigen functionals. Proofs of these lemmas are not given since they coincide with those in [6].

Lemma 1. *Let the functions $u(t,x,\lambda)$ and $H(t,x,\lambda)$ obey the conditions*

$$\chi^\pm \ne 0, \qquad \varkappa = \Delta \arg \frac{\chi^+(u)}{\chi^-(u)} = 0 \qquad (24)$$

($\Delta \arg \chi^\pm$ *is the increment of the argument of the complex function* $\chi^\pm$ *as* λ *changes from zero to unity for fixed* t *and* x). *Then, the characteristic equation* (17) *has only real roots.*

Lemma 2. *Let the functions S_1, $S_{1\lambda}$, and S_2 satisfy the Hölder condition over the variable λ and let the vector function $\boldsymbol{S}$ with the components S_1 and S_2 satisfy the relations $(\boldsymbol{F}^{1\lambda},\boldsymbol{S})=0$, $(\boldsymbol{F}^{2\lambda},\boldsymbol{S})=0$, $(\boldsymbol{F}^1,\boldsymbol{S})=0$, and $(\boldsymbol{F}^2,\boldsymbol{S})=0$. In addition, the functions $u(t,x,\lambda)$ and $H(t,x,\lambda)$ satisfy the conditions* (24). *Then, $\boldsymbol{S}\equiv 0$.*

Lemmas 1 and 2 and the definition of generalized hyperbolicity allow the following theorem to be formulated.

Theorem 1. *For flows with a velocity profile monotonic along the channel width, conditions* (24) *are necessary and sufficient for hyperbolicity of Eqs.* (7) *if the functions u, H, and ω are differentiable and the functions u_λ and ω_λ satisfy the Hölder condition over the variable λ.*

The results of analysis [9] of the characteristic properties of the shallow-water equations for shear flows with a nonmonotonic velocity profile on the vertical can be extended to the examined model (7). Let the function $u(t,x,\lambda)$ satisfy conditions (11). We define the complex function

$$\chi_1(z) = (z-u^*)\Big(1 - \frac{g}{\eta}\int_0^1 \frac{H\,d\lambda}{(u-z)^2}\Big),$$

where $u^* = u(t,x,\lambda_*)$, $u_\lambda(t,x,\lambda_*)=0$, and $u_{\lambda\lambda}(t,x,\lambda_*)\ne 0$. We assume that, for the examined smooth solution u, H, the characteristic equation $\chi_1(k)=0$ has two real roots outside the segment $[u_0,u^*]$. According to [9], the generalized hyperbolicity conditions for model (7) are formulated as follows.

Theorem 2. *For flows with a velocity profile nonmonotonic along the channel width of the class* (11), *the conditions*

$$\chi_1^\pm \ne 0, \qquad \varkappa = \Delta \arg \frac{\chi_1^+(u)}{\chi_1^-(u)} = -3\pi$$

are necessary and sufficient for hyperbolicity of Eqs. (7) *for a smooth solution $u(t,x,\lambda)$, $H(t,x,\lambda)$. The increment of the argument of the complex function is calculated on the segment $[u_0,u^*]$.*

Thus, the conditions for which Eqs. (7) are generalized hyperbolic, i.e., have only the real characteristic spectrum k^1, k^2, $[u_0,u_1]$, and the relations for the characteristics (21) are equivalent to system (7).

3. Change in the Type of the System of Equations during the Evolution of the Flow. An example of verifying the validity of the hyperbolicity conditions (24) is given below. We consider an exact solution of Eqs. (7),

$$u = (x + C(\lambda))t^{-1}, \qquad H = t^{-1}, \qquad (25)$$

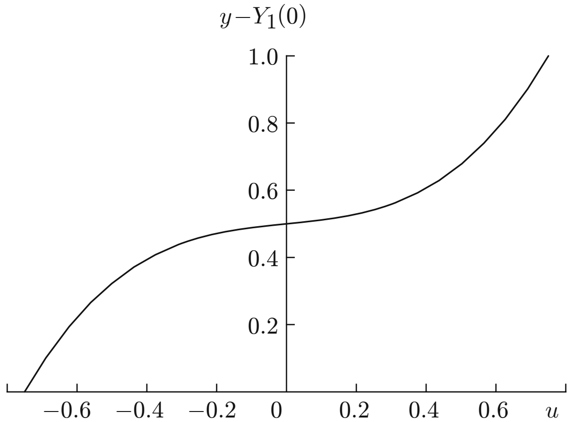

Fig. 2

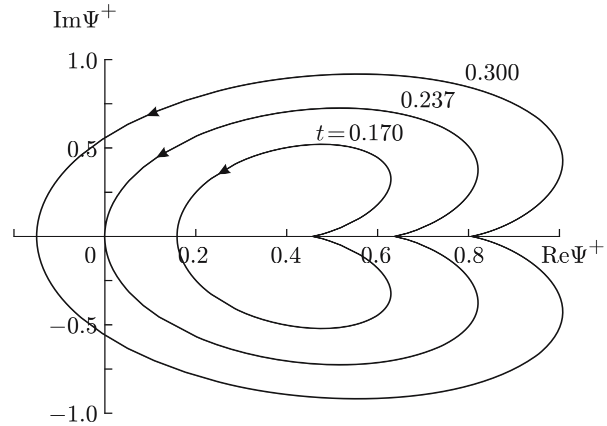

Fig. 3

Fig. 2. Velocity distribution $u(t,x,y)$ along the channel width for $t = 1$ and $x = 0$.

Fig. 3. Parametric representation of the real and imaginary parts of the function Ψ^+ (the arrows show the circulation direction).

that describes the fluid spread in a channel of constant cross section $\eta = \eta_0 = \text{const}$ under the action of pressure. Here $C(\lambda)$ is an arbitrary smooth function. In Eulerian variables, the solution has the form

$$u = \frac{x + C(y - Y_1(x))}{t}, \qquad v = \frac{x + C(y - Y_1(x))}{t}\, Y_1'(x), \qquad h = \frac{1}{t}.$$

The lateral walls of the channel are given by the equations $y = Y_1(x)$ and $y = Y_1(x) + \eta_0$, where $Y_1(x)$ is an arbitrary differentiable function.

Let us show that, for the solution considered, complex roots of the characteristic equation (17) can appear during the evolution of the flow. Let the function $C(\lambda)$ be given implicitly by the equation

$$C^3 + aC - \lambda + 1/2 = 0,$$

where a is a positive constant. This cubic equation has one real and two imaginary roots for each $\lambda \in [0, 1]$. We note that $C(\lambda)$ is a monotonically increasing function since $C'(\lambda) = (3C^2 + a)^{-1} > 0$. In addition, $C(1/2) = 0$ and $C_1 = -C_0$ [C_0 and C_1 are the values of the function $C(\lambda)$ for $\lambda = 0$ and 1, respectively]. Substitution of solution (25) into expression (23) yields

$$\chi^{\pm}(C) = 1 + 2gt\Big(\frac{(3C_1^2 + a)C_1}{C_1^2 - C^2} - 6C_1 - 3C \ln \frac{C_1 - C}{C_1 + C}\Big) \mp 6\pi gCti.$$

In this case, the functions $\chi^{\pm}$ depend only on t and $C(\lambda)$. In the verification of the hyperbolicity conditions, it is convenient to use functions $\Psi^{\pm}$ given by the formula

$$\Psi^{\pm}(C) = (C_1^2 - C^2)\chi^{\pm}(C)$$

that do not have poles at the boundary points $C = \pm C_1$.

We assume that $C_1 = 3/4$, $a = (2C_1)^{-1} - C_1^2 = 5/48$, $g = 1$, and $\eta_0 = 1$. The velocity distribution $u(t,x,y)$ along the channel width in the section $x = 0$ at the time $t = 1$ is shown in Fig. 2. Plots of the function Ψ^+ for C varying from $-C_1$ to C_1 at the times $t = 0.170$, 0.237, and 0.300 are given in Fig. 3 [the values of $\operatorname{Re}\Psi^+(C)$ are plotted on the abscissa, and the values of $\operatorname{Im}\Psi^+(C)$ on the ordinate]. The plots of the function Ψ^- are similar to those of the function Ψ^+ but the circulation direction is opposite. For $C = \pm C_1$ and $C = 0$, the imaginary part of the functions $\Psi^{\pm}$ vanish and the functions at these points take the following values:

$$\Psi^{\pm}(-C_1) = \Psi^{\pm}(C_1) = 2gt(3C_1^2 + a)C_1 = 43t/16 > 0 \qquad (t > 0),$$

$$\Psi^{\pm}(0) = C_1^2 + 2g(a - 3C_1^2)C_1 t = (3/4)(3/4 - 19t/6).$$

It is easy to see that $\Psi^{\pm}(0) > 0$ at $t \in (0, t_*)$ and $t_* = 9/38 \approx 0.237$.

From Fig. 3, it follows that, for $t = 0.17$, the increment of the argument of the functions $\Psi^{\pm}$ is equal to zero and the hyperbolicity conditions (24) are satisfied. This is valid for all $t \in (0, t_*)$. In Fig. 3, the curve constructed for $t = 0.237$ corresponds to the neutral case. For $t > t_*$, the argument of the functions $\Psi^{\pm}$ gains an increment: $\Delta \arg \Psi^+ = 2\pi$ and $\Delta \arg \Psi^- = -2\pi$. Thus, $\Delta \arg (\Psi^+/\Psi^-) = 4\pi$, which implies that the characteristic equation (17) has a complex root $k = k^3$ and a conjugate root $k = k^4 = \bar{k}^3$.

In the analysis of Eq. (17), it was shown that, at the moment of origin, the complex characteristic roots are separated from the continuous characteristic spectrum $[u_0, u_1]$. In the case considered, the complex root is separated from the middle of this segment, i.e., from the point $u_* = u(t, x, 1/2)$. The results of calculations of the roots in the channel section $x = 0$ are given below. At $t = 0.17$, there are only real characteristic roots $k^2 = -k^1 \approx 5.41$ which lie outside the segment $[u_0, u_1]$ ($u_1 = -u_0 \approx 4.41$). At $t = 0.238 > t_*$, in addition to the real roots $k^2 = -k^1 \approx 4.0647$ (in this case, $u_1 = -u_0 \approx 3.15$) there are complex roots $k^3 = -k^4 \approx 0.0046i$. With increasing time, the absolute values of the complex roots increase; in particular, at $t = 0.3$, we have $k^3 = -k^4 \approx 0.168i$.

Thus, during the evolution of the flow, the type of system (7) can change, which corresponds to long-wave instability for a certain distribution of the velocity u along the channel width.

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